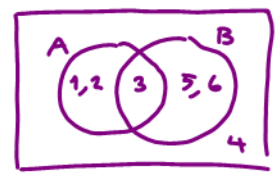
2 Review of Set Theory

Example 2.1. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$

Let $A = \{1, 2, 3\}$

$B = \{3, 5, 6\}$



2.2. Venn diagram is very useful in set theory. It is often used to portray relationships between sets. Many identities can be read out simply by examining Venn diagrams.

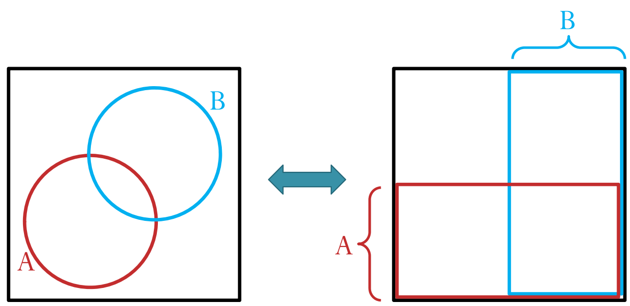


Figure 2: Example of a Venn diagram for two sets and a corresponding "K-Map"-style diagram

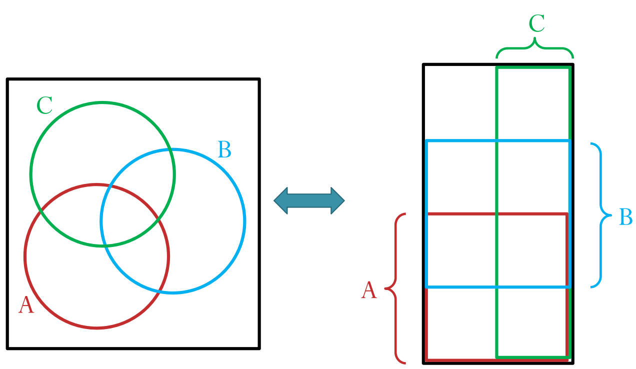


Figure 3: Example of a Venn diagram for three sets and a corresponding "K-Map"-style diagram

$2 \in A$

2.3. Membership: If ω is a member of a set A, we write $\omega \in A$.

Definition 2.4. Basic set operations (set algebra)

- Complementation: $A^c = \{\omega : \omega \notin A\}$. $A^c = \{4, 5, 6\}$
- Union: $A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}$ $A \cup B = \{1, 2, 3, 5, 6\}$
  - Here "or" is inclusive; i.e., if $\omega \in A$, we permit ω to belong either to A or to B or to both.

- Extension: The union of the events $A_1, A_2, \ldots, A_n$ is denoted by $\bigcup_{i=1}^n A_i$. It consists of all outcomes that are in ***any*** of the events A_i.

- Intersection: $A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$ $A \cap B = \{3\}$
  - Hence, $\omega \in A$ if and only if ω belongs to both A and B.
  - Extension: The intersection of the events $A_1, A_2, \ldots, A_n$ is denoted by $\bigcap_{i=1}^n A_i$. It consists of all outcomes that are in ***all*** of the events A_i.
  - $A \cap B$ is sometimes written simply as AB. We will not use that notation here.

- The ***set difference*** operation is defined by $B \setminus A = B \cap A^c$.
  - $B \setminus A$ is the set of $\omega \in B$ that do not belong to A.
  - When $A \subset B$, $B \setminus A$ is called the complement of A in B.

Reading assignment

2.5. Basic Set Identities:

- Idempotence: $(A^c)^c = A$

- Commutativity (symmetry):

$$A \cup B = B \cup A \,,\ A \cap B = B \cap A$$

- Associativity:
  - $A \cap (B \cap C) = (A \cap B) \cap C$
  - $A \cup (B \cup C) = (A \cup B) \cup C$

- Distributivity

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- **de Morgan laws**
  - $(A \cup B)^c = A^c \cap B^c$
  - $(A \cap B)^c = A^c \cup B^c$

2.6. Disjoint Sets:

empty set (null set)

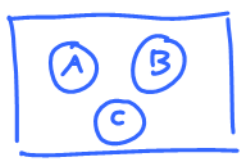
- Sets A and B are said to be ***disjoint*** ($A \perp B$) if and only if $A \cap B = \emptyset$. (They do not share member(s).)
- A collection of sets $(A_i : i \in I)$ is said to be (pairwise) ***disjoint*** or mutually exclusive [9, p. 9] if and only if $A_i \cap A_j = \emptyset$ when $i \neq j$.

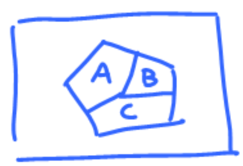
Example 2.7. Sets A, B, and C are ~~pairwise~~ disjoint if

$A \cap B = \emptyset$

$B \cap C = \emptyset$

$A \cap C = \emptyset$





2.8. For a set of sets, to avoid the repeated use of the word "set", we will call it a **collection/class/family** of sets.

$\mathcal{D} = \{A, B, C\}$

Definition 2.9. Given a set S, a collection $\Pi = (A_\alpha : \alpha \in I)$ of subsets[2] of S is said to be a ***partition*** of S if

(a) $S = \bigcup_{\alpha \in I} A_\alpha$ and The union of all the sets in the collection is S itself.

(b) For all $i \neq j$, $A_i \perp A_j$ (~~pairwise~~ disjoint). The sets in the collection are disjoint.

Remarks:

- The subsets A_α, $\alpha \in I$ are called the **parts** of the partition.

[2]In this case, the subsets are indexed or labeled by α taking values in an index or label set I

- A part of a partition may be empty, but usually there is no advantage in considering partitions with one or more empty parts.

Example 2.10. Let $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{1\}$, $B = \{3, 4\}$, $C = \{2, 5, 6\}$, and $D = \{1, 2, 5, 6\}$.

(a) The collection of sets A, B and C forms a partition of set S.

(b) Another partition is the collection of sets B and D.

Example 2.11 (Slide:maps)**.**

Example 2.12. Let E be the set of students taking ECS315

A = set of students who will get A

B+ = " " B+

B

C+

C

D+

? ∅ = D

? ∅ = F

W

The collection $\{A, B+, B, \ldots, F, W\}$ is a partition of E

Definition 2.13. Important sets involving (real) numbers:

(a) The set $\mathbb{N}$ of all natural numbers.

- More specifically, $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- Note that ∞ is not a member of this set.

(b) The set $\mathbb{Z}$ of all integers

(c) The set $\mathbb{R}$ of all real numbers

- $\mathbb{R}$ can be expressed as an interval $(-\infty, \infty)$.

(d) An **interval** is a set of real numbers with the property that any number that lies between two numbers in the set is also included in the set. The interval of numbers between a and b, including a and b, is often denoted $[a, b]$. The two numbers are called the **endpoints** of the interval.

"(" ")" } exclude ...

To indicate that one of the endpoints is to be excluded from the set, the corresponding square bracket can be replaced with a parenthesis. For example,

$$[a, b) = \{x \in \mathbb{R} \,|\, a \leq x < b\}.$$

"[" "]" } include the corresponding boundary number

Definition 2.14. A ***singleton*** is a set with exactly one element.

- Ex. $\{1.5\}$, $\{.8\}$, $\{\pi\}$. $\{apple\}$, $\{a\}$
- *Caution*: Be sure you understand the difference between the outcome -8 and the event $\{-8\}$, which is the set consisting of the single outcome -8.

Definition 2.15. The ***cardinality*** (or size) of a collection or set A, denoted $|A|$, is the number of elements of the collection. This number may be finite or infinite.

A is a finite set iff $|A| \in \mathbb{N} \cup \{0\}$

(a) A **finite** set is a set that has a finite number of elements. In other words, it is either

  (i) an empty set,

  (ii) a singleton, or

  (iii) a set whose elements can be listed in the form $\{a_1, a_2, \ldots, a_n\}$ for some $n \in \mathbb{N}$.

(b) A set that is not finite is called **infinite**. These sets have more than n elements for any integer n.

Definition 2.16. A **countable** set is a set with the same cardinality as some subset of the set of natural numbers. A countable set is either

$\emptyset$ $\{a_1, a_2, \dots, a_n\}$

(a) a finite set (potentially an empty set), or

(b) an infinite set if its elements can be listed in a sequence: $a_1, a_2, \dots$. In such case, the set is said to be **countably infinite**.

infinite + countable = countably infinite

Whether finite or infinite, the elements of a countable set can always be counted one at a time and, although the counting may never finish, every element of the set is associated with a natural number. Countable sets form the foundation of a branch of mathematics called discrete mathematics.

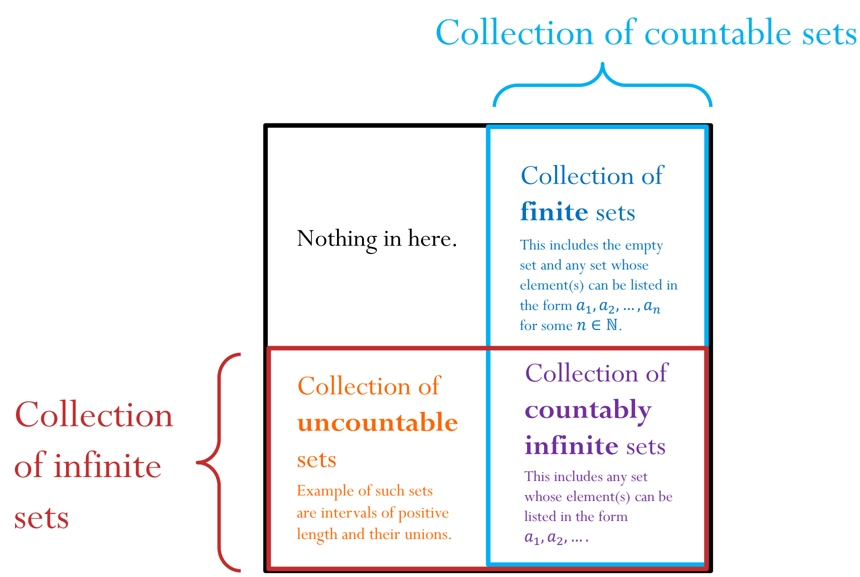


Figure 4: Categorizing sets by whether they are infinite and whether they are countable.

Example 2.17. Examples of countably infinite sets:

- the set $\mathbb{N} = \{1, 2, 3, \dots\}$ of natural numbers, $a_1\ a_2\ a_3$ $a_k = k$
- the set $\{2k : k \in \mathbb{N}\}$ of all positive even numbers, $= \{2, 4, 6, 8, \dots\}$ $a_1\ a_2\ a_3\ a_4$ $a_{50} = 100$ $a_{101} = 202$
- the set $\{2k - 1 : k \in \mathbb{N}\}$ of all positive odd numbers, $= \{1, 3, 5, \dots\}$ $a_1\ a_2\ a_3$

$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

- the set $\mathbb{Z}$ of integers, $= \{0, \pm 1, \pm 2, \pm 3, \ldots\} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$

a_1 a_2 a_3 a_4 a_5

$a_{100} = 50$

$a_k = -315 \quad k = 631$

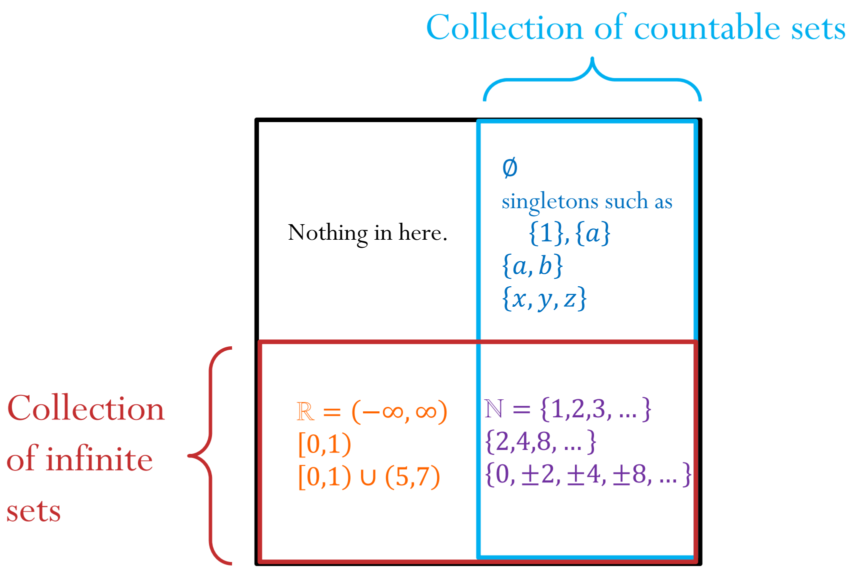


Figure 5: Examples of Infinite Sets and Countable Sets

Definition 2.18. A set that is not countable is called **uncountable** set (or uncountably infinite set). It contains too many elements to be countable.

Example 2.19. Example of uncountable sets[3]:

- $\mathbb{R} = (-\infty, \infty)$
- interval with positive length: $[0, 1]$
- union of intervals with positive length: $(2, 3) \cup [5, 7)$

sets: finite; infinite

infinite: countably infinite } countable (discrete) (digital)

uncountable (continuous) (analog)

[3]We use a technique called diagonal argument to prove that a set is not countable and hence uncountable.

Set Theory	Probability Theory
Set	Event
Universal set	Sample Space (Ω)
Element	Outcome (ω)

Table 1: The terminology of set theory and probability theory

	Event Language
A	A occurs
A^c	A does not occur
$A \cup B$	Either A or B occur
$A \cap B$	Both A and B occur

Table 2: Event Language

$A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \equiv$ at least one of the events occurs

$A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n \equiv$ all events happen/occur

2.20. From Definitions 2.15 and 2.16, and 2.18, we can categorize sets according to whether they are infinite and whether they are countable. This is illustrated in Figure 4.

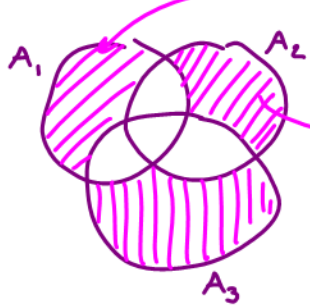
Definition 2.21. Probability theory renames some of the terminology in set theory. See Table 1 and Table 2.

- Sometimes, ω's are called states, and Ω is called the state space.

2.22. Because of the mathematics required to determine probabilities, probabilistic methods are divided into two distinct types, discrete and continuous. A discrete approach is used when the number of experimental outcomes is finite (or infinite but countable). A continuous approach is used when the outcomes are continuous (and therefore infinite). It will be important to keep in mind which case is under consideration since otherwise, certain paradoxes may result.

Consider three events A_1, A_2, A_3

Let B be the event that exactly one of the events above occurs.



$B = (A_1 \cap A_2^c \cap A_3^c) \cup (A_1^c \cap A_2 \cap A_3^c) \cup (A_1^c \cap A_2^c \cap A_3)$

In general, when we have $A_1, A_2, A_3, \ldots, A_m$, the event that exactly one of them occurs is

$$\bigcup_k \left(A_k \cap \left(\bigcap_{i \neq k} A_i^c \right) \right)$$

$$\cancel{= \bigcup_{k=1}^{m} \left(\underbrace{\left(\bigcap_{i=1}^{k-1} A_i^c \right)}_{\text{problematic when } k=1} \cap A_k \cap \left(\bigcap_{i=k+1}^{m} A_i^c \right) \right)}$$